A RENORMING OF DUAL SPACES

BY

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ABSTRACT

If Banach spaces X, X^* are both weakly compactly generated, then X has an equivalent norm whose dual on X^* is locally uniformly rotund.

1. Introduction

The proof of the main result given in Section 3 is done exactly by the same method as Trojanski's [10] proof for generally non-dual (WCG) Banach spaces, the only new point here is the arrangement that all cases are w^* lower semicontinuous in X^*.

We will work in real Banach spaces. A Banach space X (in short, a B-space X) is weakly compactly generated (WCG) if X is the closed linear hull of some weakly compact absolutely convex $K \subset X$, i.e., $X = sp\ K$. A B-space X is locally uniformly rotund (LUR) if the relations $\| x_n \| = \| x \| = 1$, $\lim \| x_n + x \| = 2$ imply $\lim \| x_n - x \| = 0$. Furthermore, a B-space X is an (F) space if the norm of X is Fréchet differentiable at any nonzero point. $c_0(\Gamma)$ is the B-space of all real valued functions f on a set Γ such that for any $\varepsilon > 0$, $\{\gamma \in \Gamma; |f(\gamma)| > \varepsilon\}$ is finite, with the supremum norm. For a B-space X, dens X is the smallest cardinal number of a norm dense subset of X.

2. Applications of the main result

The following corollary solves problem 13 of [8].

COROLLARY 1. *If X and X^* are (WCG), then X has an equivalent (LUR) and (F) norm whose dual is also (LUR).*

PROOF. The result follows from the Asplund's averaging procedure [3] and from the duality between (F) and (LUR) [9].

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COROLLARY 2. *If X, X^* are both (WCG) then X is an (SDS) space in the sense of* [4], *p.* 31, *i.e., any continuous convex function in X is Fréchet differentiable on a dense G_δ subset of its domain of continuity.*

PROOF. See [4], p. 32.

The next corollary solves partially problem 16 in [8].

COROLLARY 3. *If X, X^* are both (WCG), then X^* is a boundedly Krejn-compact B-space in the sense of* [5], p. 1, *i.e., each norm closed convex bounded subset of X^* is the norm closed convex hull of its extreme points.*

PROOF. See [5], p. 4 or [11], p. 453.

COROLLARY 4. *If X^*, X^{**} are both (WCG), then any closed convex bounded subset of X is the closed convex hull of its strongly exposed points (see* [7] *for definition).*

PROOF. Use [11], p. 452.

3. Preparatory lemmas

We will need some modifications of the ideas of [1]. We state them in several lemmas.

LEMMA 1. *Let X be a linear space with 3 norms $|\cdot|_1, |\cdot|_2, |\cdot|_3$, such that $|x|_1 \leqq |x|_2, |x|_1 \leqq |x|_3$ for every $x \in X$. Then, given $\varepsilon > 0$, an integer $n > 0$, l elements $f_1, \cdots, f_l \in (X, |\cdot|_2)^*$, and a finite-dimensional subspace $B \subset X$, there exists an $\aleph_0$-dimensional subspace $C \subset X$ containing B, such that for every subspace Z of X with $Z \supset B$ and $\dim Z/B = n$, there is a linear operator $T: Z \to C$ with $|T|_\alpha \leqq 1 + \varepsilon$ (for all $\alpha = 1,2,3$), $Tb = b$ for every $b \in B$ and $|f_k(z) - f_k(Tz)| \leqq \varepsilon |z|_2$ for every $z \in Z$ and $k = 1, \cdots, l$.*

PROOF. Let P be a $|\cdot|_1$-bounded projection of X onto B. Then P is bounded in all three norms and let K be such that $|P|_\alpha \leqq K$, for $\alpha = 1,2,3$. Choose $m > 1$ such that $m > 6(1 + K)\varepsilon^{-1}$.

Let r be an integer. Choose $b_1, \cdots, b_p$ in B such that, for every $b \in B$, and every $\alpha = 1,2,3$ the following holds: if $|b|_\alpha \leqq r$ then there is an h ($1 \leqq h \leqq p$) such that such that $|b - b_h|_\alpha < m^{-1}$.

Let us consider the norm $|\lambda| = \sum_{i=1}^n |\lambda_i|$ in the Euclidean space R^n. Let s be an integer and choose the elements $\lambda^1, \cdots, \lambda^q$ in the unit sphere

$S^n = \{\lambda \in R^n; |\lambda| = 1\}$ of R^n, such that for every $\lambda \in S^n$ there is j, $1 \leqq j \leqq q$, so that $|\lambda - \lambda^j < m^{-1}s^{-1}$.

Fix now the integers r and s and define the following $N = 3n + 3pq + ln$ real-valued functions of $x = (x_1, \cdots, x_n) \in X^n$,

$$|x_i|_\alpha, \ |b_h + \sum_{i=1}^{n} \lambda_i^j x_i|_\alpha, \qquad f_k(x_i)$$

$1 \leqq i \leqq n$, $1 \leqq \alpha \leqq 3$, $1 \leqq h \leqq p$, $1 \leqq j \leqq q$, $1 \leqq k \leqq l$.

These functions can be regarded as a function ϕ from X^n into R^N. Taking, in R^N, the metric ρ of maximal coordinate distance, we choose a sequence $\{\phi(x^t)\}_t$, $x^t = (x_1^t, \cdots, x_n^t) \in X^n$ which is ρ-dense in $\phi(X^n)$. This sequence is constructed for fixed r, s. Thus we have a sequence $\{x^t\} = \{x^{trs}\}$ for each r, s. Let C be the subspace spanned by B and the $\{x_i^{trs}\}$ $(i = 1, \cdots, n;\ t, r, s = 1, 2, \cdots)$.

Given any $Z \supset B$ with $\dim Z/B = n$, choose $z_1, \cdots, z_n \in (I - P)Z$, such that $|\Sigma \lambda_i z_i|_\alpha \geqq |\lambda|$ for every $\lambda \in R^n$ and every $\alpha = 1, 2, 3$. (It is sufficient to choose $z_1, \cdots, z_n$ linearly independent and multiply them all by a sufficiently large number.) Choose s such that $|z_i|_\alpha \leqq s$ for all $1 \leqq i \leqq n$, $\alpha = 1, 2, 3$, and choose r such that $2s + 1 < \varepsilon(r - s)$. Let us now fix these values of s and r for the rest of the proof.

Let $x = (x_1, \cdots, x_n)$ be an element from the above constructed sequence (for s and r chosen) such that $\rho(\phi(x_1, \cdots, x_n), \phi(z_1, \cdots, z_n)) < m^{-1}$. Define on Z

$$T\left(b + \sum_{i=1}^{n} \lambda_i z_i\right) = b + \sum_{i=1}^{n} \lambda_i x_i \quad \text{where } b \in B.$$

Obviously $Tz \in C$ and $Tb = b$ for all $b \in B$. To prove that $|T|_\alpha \leqq 1 + \varepsilon$, it suffices to show that $|b + \Sigma \lambda_i x_i|_\alpha \leqq (1 + \varepsilon)|b + \Sigma \lambda_i z_i|_\alpha$ whenever $|\lambda| = 1$.

If $|b|_\alpha \geqq r$, then $|b + \Sigma \lambda_i z_i|_\alpha \geqq r - s$, while

$$\begin{aligned} |b + \Sigma \lambda_i x_i|_\alpha &\leqq |b + \Sigma \lambda_i z_i|_\alpha + |\Sigma \lambda_i z_i|_\alpha + |\Sigma \lambda_i x_i|_\alpha \\ &\leqq |b + \Sigma \lambda_i z_i|_\alpha + s + (s + 1) \leqq |b + \Sigma \lambda_i z_i|_\alpha + \varepsilon(r - s) \\ &\leqq (1 + \varepsilon)|b + \Sigma \lambda_i z_i|_\alpha. \end{aligned}$$

(We used the fact that $||x_i|_\alpha - |z_i|_\alpha| \leqq m^{-1} \leqq 1$; hence $|x_i|_\alpha \leqq |z_i|_\alpha + 1 \leqq s + 1$.)

If $|b|_\alpha \leqq r$, let b_h be m^{-1}-approximation to b and let λ^j be $m^{-1}s^{-1}$-approximation to $\lambda \in S^n$. We have

$$\begin{aligned} |b + \Sigma \lambda_i x_i|_\alpha - |b + \Sigma \lambda_i z_.|_\alpha &\leqq 2|b - b_h|_\alpha \\ &+ |b_h + \sum_i \lambda_i^j x_i|_\alpha - |b_h - \sum_i \lambda_i^j z_i|_\alpha \end{aligned}$$

$$+ \left| \sum_i (\lambda_i^j - \lambda_i)x_i \right|_\alpha + \left| \sum_i (\lambda_i^j - \lambda_i)z_i \right|_\alpha$$

$$\leqq 2m^{-1} + m^{-1} + (s+1)m^{-1}s^{-1} + sm^{-1}s^{-1} \leqq 6m^{-1},$$

while

$$\varepsilon \left| b + \Sigma\lambda_i z_i \right|_\alpha \geqq \varepsilon \left| I - P \right|_\alpha^{-1} \left| \Sigma\lambda_i z_i \right|_\alpha \geqq \varepsilon(1+k)^{-1} > 6m^{-1}.$$

If $z = b + \Sigma\lambda_i z_i$, then

$$\left| f_k(z) - f_k(Tz) \right| = \left| \Sigma\lambda_i(f_k(z_i) - f_k(x_i)) \right| \leqq m^{-1}\left| \lambda \right|,$$

while

$$\left| z \right|_2 \geqq \left| I - P \right|_2^{-1} \left| \Sigma\lambda_i z_i \right|_2 \geqq (1+K)^{-1} \left| \lambda \right|.$$

Hence

$$\left| f_k(z) - f_k(Tz) \right| / \left| z \right|_2 \leqq m^{-1}(1+K) < \varepsilon.$$

As in [1], if there are given some norms on X, all topological terms will refer to the $\| \cdot \|$-norm.

Similarly as in [1] we prove

LEMMA 2. *Let X be a linear space with three norms $| \cdot |$, $\| \cdot \|$, $||| \cdot |||$ such that the $||| \cdot |||$-unit ball is $\| \cdot \|$-weakly compact. Suppose that the $| \cdot |$-topology is weaker than the $\| \cdot \|$-topology on X. Let μ be the first ordinal of cardinality dens X and let $\{x_\alpha, \alpha < \mu\}$ be dense in X. Then there is a "long sequence" of linear projections $\{P_\alpha, \omega \leqq \alpha \leqq \mu\}$ such that $| P_\alpha | = \| P_\alpha \| = ||| P_\alpha ||| = 1$, $x_\alpha \in P_{\alpha+1}X$, dens $P_\alpha X \leqq \bar{\bar{\alpha}}$ for every α, $P_\alpha P_\beta = P_\beta P_\alpha = P_\beta$ if $\beta < \alpha$, $P_\alpha x \in sp\{P_{\xi+1}x\}_{\xi<\alpha}$, and $\{\alpha; \| P_{\alpha+1}x - P_\alpha x \| > \varepsilon\}$ is finite for any $x \in X$, $\varepsilon > 0$.*

Now we are able to prove

LEMMA 3. *Assume X is a B-space such that X, X^* are both (WCG). Let $\| \cdot \|$ denote the natural norm on X^* and let μ be the first ordinal of cardinality dens X^*. Then there is a dense subset $\{f_\alpha; \alpha < \mu\}$ in X^*, and a "long sequence" $\{P_\alpha; \omega \leqq \alpha \leqq \mu\}$ of linear projections on X^* such that $\| P_\alpha \| = 1$, P_α is w^*–w^* continuous on X^*, $f_\alpha \in P_{\alpha+1}X^*$, dens $P_\alpha X^* \leqq \bar{\bar{\alpha}}$ for any α, $P_\alpha P_\beta = P_\beta P_\alpha = P_\beta$ for $\beta < \alpha$, $P_\alpha f \in sp\{P_{\xi+1}f\}_{\xi<\alpha}$ and $\{\alpha; \| P_{\alpha+1}f - P_\alpha f \| > \varepsilon\}$ is finite for every $f \in X^*$ and $\varepsilon > 0$.*

PROOF. Let K, L be weakly compact absolutely convex sets in X, X^* respectively, such that $X = \text{sp } K$, $X^* = \text{sp } L$. Let us define on X^* another norm

$|f| = \sup\{|f(x)|; x \in K\}$ and on the linear hull Y of L in X^* the third norm $|||f||| = \inf\{\lambda > 0; f \in \lambda L\}$.

Choose $\{f_\alpha; \alpha < \mu\}$, a $\|\cdot\|$-dense net in Y. By Lemma 2, there is a long sequence of linear projections $\tilde{P}_\alpha$ on Y with the properties stated in Lemma 2. Let us extend $\tilde{P}_\alpha$ $\|\cdot\|$-continuously to projection $P_\alpha: X^* \to X^*$. Obviously $\|P_\alpha\| = 1$ and P_α satisfy the properties of Lemma 2. It can easily be shown that for every $\varepsilon > 0$ the set $\{\alpha; \|P_{\alpha+1}f - P_\alpha f\| > \varepsilon\}$ is finite even for every $f \in X^*$.

It remains to show w^*–w^* continuity of P_α. For this, observe that the P_α's are $|\cdot|$-continuous. In fact, denote by $\hat{Y}$ the completion of $(Y, |\cdot|)$ and by $\hat{P}_\alpha$ the $|\cdot|$-continuous extention of $\tilde{P}_\alpha$ on $\hat{Y}$. Obviously, $X^* \subset \hat{Y}$. Now it is easy to see that $P_\alpha = \hat{P}_\alpha$ on X^* and thus the P_α's are $|\cdot|$-continuous and $|P_\alpha| = 1$.

Now using the fact that K is weakly compact convex we conclude, exactly as in the proof of Proposition 2 of [1], that the identity mapping of X^* is w^*–w continuous, where w means the weak topology of the norm $|\cdot|$. The $\|\cdot\|$-unit ball B of X^* being w^*-compact, we see that the w and w^* topologies coincide on B. As every P_α is w–w continuous and $P_\alpha B \subset B$, it follows that the P_α's are w^*–w^* continuous on B and in virtue of the Banach-Dieudonné's theorem the P_α's are w^*–w^* continuous on X^*.

Like S. Trojanski ([10], p. 177) we will need the following.

LEMMA 4. *Suppose a Banach space X and its dual X^* are both (WCG). Then there is a long sequence of bounded linear w^*–w^* continuous operators $T_\alpha: X^* \to X^*$ $(\alpha \in \Lambda)$ such that*

(i) *for any $f \in X^*$ and $\varepsilon > 0$, the set*

$$\Lambda(f, \varepsilon) = \{\alpha; \|T_{\alpha+1}f - T_\alpha f\| > \varepsilon(\|T_\alpha\| + \|T_{\alpha+1}\|)\}$$

is finite, where $\|\cdot\|$ is the natural norm on X^*,

(ii) *for any $f \in X^*$,*

$$f \in Y_f \equiv \mathrm{sp}[\|T_1 f\| T_1 X^* \cup \bigcup_{\alpha \in \Lambda(f)} (T_{\alpha+1} - T_\alpha)X^*],$$

where $\Lambda(f) = \cup_{\varepsilon > 0} \Lambda(f, \varepsilon)$,

(iii) *dens sp $(T_{\alpha+1} - T_\alpha)X^* \leqq$ dens $T_1 X^* = \aleph_0$.*

The proof follows the Trojanski's proof ([10], p. 177), by observing that if P_γ^* are $w^* - w^*$ continuous projections on X^*, then $(P_{\gamma+1}^* - P_\gamma^*)X^*$ is isometrically isomorphic to $((P_{\gamma+1} - P_\gamma)X)^*$ by a mapping which is w^*–w^* continuous with its inverse and both $(P_{\gamma+1}^* - P_\gamma^*)X^*$ and $(P_{\gamma+1} - P_\gamma)X$ are (WCG).

For the proof of our theorem we will need the following observation.

LEMMA 5. *If X is a Banach space and $L \subset X$ is a closed subspace of it, then the distance of $f \in X^*$ from $L^{\perp}$ is a w^* lower semicontinuous functional.*

PROOF. If R denotes the restriction of f to L, then it is easily seen that

$$\rho(f, L^{\perp}) = \|f\|_{X^*/L^{\perp}} = \ Rf\|_{L^*}.$$

4. Proof of the main result

The proof follows exactly as the proof of Trojanski ([10], p. 175, 176) and all cases can be made, by use of the lemmas above, to be w^* lower semi-continuous. For example the function $E_A^{(n)}(f)$ on X^* defined on p. 175 is w^* lower semi-continuous because it is the distance from a finite dimensional subspace of X^*. And for an operator $T: X^* \to c_0(\Gamma)$, one should take the operator given by proposition 2 of [1] which is $w^* - w$ continuous.

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